Theory of two-dimensional mean field electron magnetohydrodynamics

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Theoretical studies of mean field electrodynamics for diffusive processes in the electron magnetohydrodynamic (EMHD) model is presented. In contrast to magnetohydrodynamics (MHD), the evolution of the magnetic field here is governed by a nonlinear equation in magnetic field variables. A detailed description of diffusive processes in two dimensions are presented in this paper. In particular, it has been shown analytically that the turbulent magnetic field diffusivity is suppressed from naive quasilinear estimates. It is shown that for complete whistlerization of the spectrum, turbulent diffusivity vanishes. The question of whistlerization of the turbulent spectrum is investigated numerically, and a reasonable tendency towards whistlerization is observed. Numerical studies also show suppression of magnetic field diffusivity in accordance with analytical estimates.
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I. INTRODUCTION

Transport and amplification properties of a large-scale magnetic field remains an area of active investigation. This is primarily due to its relevance in a variety of physical phenomena. For example, the existence of the magnetic field in the universe is understood on the basis of an amplification process by some kind of dynamo mechanism. Another interesting phenomenon is the release of high-energy bursts in solar flares, etc.[1] This is believed to occur as a result of the reconnection of magnetic fields, which can happen in the presence of finite diffusivity. However, there is only modest quantitative understanding of these processes. The amount of magnetic energy released by reconnection depends on the value of diffusivity, which turns out to be too small to provide an explanation of the vast energy released in these bursts. There have been attempts then to understand these phenomenon on the basis of turbulent magnetic field diffusivity, which is directly related to the question of transport of a large-scale magnetic field in the presence of turbulence.[2–5] Most theories put forward in these areas are cast within the magnetohydrodynamic (MHD) system. Lately, however, there has been some work which makes use of models pertaining to faster time scales.[6] It is on this regime that we are going to focus here.

In this work we address the question of diffusion of a long-scale magnetic field in the presence of small-scale turbulent magnetic fluctuation ocurring at time scales which are faster than the ion response time. For such phenomena, the evolution of the magnetic field is governed by electron flow velocity. Ions being stationary, the flow velocity of electrons determines current and hence is directly related to the curl of magnetic field. Thus unlike MHD, in this approximation, heretofore referred as electron magnetohydrodynamic (EMHD) approximation, the magnetic field itself evolves through an explicitly nonlinear equation. This should be contrasted with the MHD model in which nonlinear effects creep indirectly through the Lorentz force operating on the plasma flow.

This paper is organized as follows. In Sec. II we present the salient features of the electron magnetohydrodynamics (EMHD) model. In Sec. III we study the evolution of the mean magnetic field in two dimensions within the framework of the EMHD description. In two dimensions there is no amplification of the large-scale field; it can only diffuse. We obtain an expression for the effective diffusion coefficient and show that it is suppressed from the naive quasilinear estimates. For complete whistlerization, i.e., when turbulence is comprised only of randomly interacting whistler waves (whistler modes being the normal modes of EMHD model), we show that there is no turbulent contribution to diffusivity. This then raises the pertinent question about the constituents of turbulent state in this particular model. It becomes important to know whether the turbulent state is comprised entirely of randomly interacting whistler waves, or merely a collection of random eddies, or a combination of both whistlers and eddies represent the true scenario. We address these question in Sec. IV by numerically simulating the case of decaying turbulence for EMHD equations. The initial condition is chosen to be random, i.e., no whistlers to begin with. The study of the final state reveals evidence of whistlerization. In Sec. V we numerically investigate the problem of diffusion, which shows suppression of magnetic field diffusivity, essentially confirming our analytical findings of Sec. III. Section VI contains discussion and conclusion.

II. THE MODEL

Electron magnetohydrodynamics (EMHD) is the theory to describe the motion of a magnetized electron fluid in the presence of self-consistent and external electric and magnetic fields. Such a theory is applicable when time scales of interest are fast (e.g., lying between electron and ion gyrofrequencies) so that ions being massive and unmagnetized play a passive role as a neutralizing background, and the dominant

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role in dynamics is played by a strongly magnetized electron species. Phenomena having such time scales are often encountered in a number of plasma operated devices (e.g., switches, focusing devices, fast Z-pinches, etc.).[7] Moreover, in the description of collisionless magnetic reconnection[8,6] as well as in certain problems related to ionosphere, EMHD paradigm is invoked frequently. The entire whistler physics is premised on the EMHD regime of dynamics.

The EMHD model is obtained by using (i) the electron momentum equation; (ii) current expressed in terms of electron velocity $\mathbf{J}=-en_e v_e$ as ions are stationary at fast time scales depicted by this model; and (iii) Ampere's law, where displacement current is ignored under the assumption ($\omega \ll \omega_{pe}^2/\omega_{ce}$). The magnetic field then evolves according to the following equation:

$$\frac{\partial}{\partial t}(\nabla\times\mathbf{P})=\nabla\times(\mathbf{v}_e\times(\nabla\times\mathbf{P}))-m_e\nu\nabla\times\mathbf{v}_e . \tag{1}$$

Here m_e and $\mathbf{v}_e$ are electron mass and velocity, respectively, $\mathbf{P}$ is canonical momenta defined as $\mathbf{P}=m_e\mathbf{v}_e-e\mathbf{A}/c$ ($\mathbf{A}$ being vector potential of the magnetic field), and ν represents electron ion collision frequency. Using the relationship between current and electron velocity we obtain $\nabla\times\mathbf{P}=e(d_e^2\nabla^2\mathbf{B}-\mathbf{B})/c$, where $d_e=c/\omega_{pe}$ is the skin depth.

It is clear from Eq. (1) that $(d_e^2\nabla^2\mathbf{B}-\mathbf{B})$ is frozen in the electron fluid flow. In the limit when electron inertia can be ignored, magnetic field lines are carried along with the electron fluid. Since $v_e\sim-\nabla\times\mathbf{B}$, the evolution equation for the magnetic field is nonlinear in $\mathbf{B}$. This can be contrasted with the MHD model where the magnetic field evolution is governed by an equation which is intrinsically linear in $\mathbf{B}$. In MHD, the nonlinear effects then arise as a result of back reaction on fluid flow through Lorentz force terms. Basically, in EMHD $\mathbf{v}_e\sim-\nabla\times\mathbf{B}$, and so electron flow is directly related to the instantaneous magnetic field, whereas in MHD the evolution of flow velocity $\mathbf{v}$ depends on the magnetic field through Lorentz force term and hence $\mathbf{v}$ has a memory of the past magnetic field configuration. The MHD model is applicable for scale lengths which are longer than the ion skin depth. The EMHD, on the other hand, depicts phenomenon having scale lengths shorter than the ion skin depth. Another distinction from MHD arises due to the presence of an intrinsic scale, viz., electron skin depth $d_e=c/\omega_{pe}$ in the EMHD model, which separates the two regimes: one in which electron inertia is important, and the other where electron inertia plays no role. The character of EMHD evolution equations changes in these two disparate regimes of scale lengths.

In two dimensions (i.e., when variations are confined in $x-y$ plane) Eq. (1) can be simplified and cast in terms of two scalar variables ψ and b, which define the total magnetic field by the expression $\mathbf{B}=\hat{z}\times\nabla\psi+b\hat{z}$. The following coupled set then represents the evolution of these scalar variables:

$$\frac{\partial}{\partial t}(\psi-\nabla^2\psi)+\hat{z}\times\nabla b\cdot\nabla(\psi-\nabla^2\psi)=\eta\nabla^2\psi, \tag{2}$$

$$\frac{\partial}{\partial t}(b-\nabla^2 b)-\hat{z}\times\nabla b\cdot\nabla\nabla^2 b+\hat{z}\times\nabla\psi\cdot\nabla\nabla^2\psi=\eta\nabla^2 b. \tag{3}$$

Here we have chosen to normalize length by electron skin depth $d_e=c/\omega_{pe}$, magnetic field by a typical amplitude B_0, and time by the corresponding electron gyrofrequency. In the nonresistive limit, the above coupled equations support the following quadratic invariants,

$$E=\frac{1}{2}\int[(\nabla\psi)^2+b^2+(\nabla^2\psi)^2+(\Delta b)^2]\,dxdy,$$

which represents total energy (sum of the magnetic and the kinetic energy),

$$H=\int(\psi-\nabla^2\psi)^2\,dxdy,$$

the mean square magnetic potential, and

$$K=\int(\psi-\nabla^2\psi)(b-\nabla^2 b)\,dxdy,$$

the cross helicity. The fields b and ψ are chosen to be uncorrelated initially in our numerical simulations. On the basis of the existence of these quadratic invariants it can be inferred that the mean square magnetic potential cascades towards longer scale. We will be making use of this later in our derivation for turbulent diffusivity.

Linearizing the evolution equations in the presence of the uniform magnetic field B_0 pointing in the y direction leads to following dispersion relation

$$\omega=\pm\frac{kk_y d_e^2\omega_{ci}}{(1+k^2 d_e^2)}$$

for whistlers, the normal mode of oscillations in the EMHD regime. It is clear from the dispersion relation that the propagation of these waves is preferentially parallel to the magnetic field. Furthermore, whistler wave excitations lead to a coupling of the form $b_k=\pm k\psi_k$ between the two perturbed fields. This relation between perturbed fields then leads to an equipartition between energy associated with poloidal and axial fields. An initial unequal distribution of energy in the poloidal and axial fields ultimately has a tendency towards redistribution and achieving equipartition as a result of the whistlerization of the spectrum. It is observed that time asymptotically the turbulent state in EMHD consists of a gas of whistlers interspersed with a collection of random eddies.

There has been considerable interest lately to understand the features of EMHD turbulence both in two and three dimensions in terms of power spectra and cascade properties of square invariants supported by the model.[9] Our attempt here, however, is to understand the role of EMHD turbulence in determining the diffusion of long-scale magnetic field.

III. SUPPRESSION OF TURBULENT MAGNETIC DIFFUSIVITY IN 2D

In this section we concentrate on the transport of magnetic field in two dimension. In 2D the magnetic field can only diffuse, thus our endeavour here is to estimate the effective magnetic diffusivity in the presence of turbulence.

We will concentrate here on turbulent scale lengths longer than electron skin depth. In this regime of scale lengths, i.e., for $kd_e \ll 1$ electron inertia effects are unimportant and as mentioned in earlier section magnetic field lines are frozen in electron fluid flow. Thus turbulence in electron velocity leads to the diffusion of magnetic flux. This diffusion of magnetic field lines, arising as a result of turbulence and not due to resistivity, is termed as turbulent diffusivity of the magnetic field. The effective turbulent diffusivity would thus depend on electron fluid flow velocity. A naive quasilinear estimate would thus predict that magnetic field diffusivity $\beta \sim \tau v_e^2 \sim \tau(\nabla b)^2$, where τ is some averaged correlation time for electron flow velocity $v_e = \hat{z} \times \nabla b$ in $x-y$ plane, and b is the z component of turbulent small scale magnetic field. This suggests that magnetic field diffusion in the $x-y$ plane is solely determined by turbulent properties of the z (i.e., the axial) component of the magnetic field. However, this does not represent the complete picture. We will now show that the presence of small-scale turbulence in the poloidal magnetic field results in the suppression of such estimates of diffusivity. This is similar to the work carried out by Gruzinov,[10] Cattaneo,[11] and others in the context of MHD. In MHD magnetic field lines are tied to plasma flow velocity. It is observed that magnetic field diffusivity is suppressed from the quasilinear estimates given solely in terms of plasma flow velocity. The presence of small-scale turbulence in the magnetic field, which opposes the fluid motion through $\mathbf{J} \times \mathbf{B}$ backreaction, is found to be responsible for such a suppression.

We choose to represent the small-scale turbulence in fields b and ψ as

$$b(x,t) = \sum_k b_k(t) \exp(i\mathbf{k} \cdot \mathbf{r}),$$

$$\psi(x,t) = \sum_k \psi_k(t) \exp(i\mathbf{k} \cdot \mathbf{r}).$$

In addition to this, we assume the existence of a large-scale magnetic field pointing along the y direction characterized by magnetic stream function of the following form

$$\psi_0 = \psi_q \exp(iq_x x) + \text{c.c.}$$

This magnetic field has a scale length $q^{-1} \gg k^{-1}$ and, hence, when considering averaging over the scale of turbulence, this field can be essentially treated as constant in space. We are interested in understanding the process of diffusion of this long-scale field in the presence of small-scale turbulence in variables b and ψ, i.e., we seek an equation of the kind

$$\frac{\partial \psi_q}{\partial t} = -\beta q_x^2 \psi_q, \tag{4}$$

and are interested in determining β in terms of the properties of small-scale turbulence. The qth Fourier component of Eq. (2) yields

$$(1+q_x^2)\frac{d\psi_q}{dt} + \langle \hat{z} \times \nabla b \cdot \nabla(\psi - \nabla^2\psi)\rangle_q = -\eta q_x^2 \psi_q. \tag{5}$$

Here the second term signifies generation of qth mode as a result of nonlinear coupling between high-k turbulent fields. The angular brackets indicate an ensemble average. The above equation can be rewritten as

$$(1+q_x^2)\frac{d\psi_q}{dt} + i\mathbf{q} \cdot \langle \hat{z} \times \nabla b(\psi - \nabla^2\psi)\rangle_q = -\eta q_x^2 \psi_q.$$

We denote $\langle \hat{z} \times \nabla b(\psi - \nabla^2\psi)\rangle_q$ by $\boldsymbol{\Gamma}$ representing the nonlinear flux. Since $q_y = 0$, $i\mathbf{q} \cdot \boldsymbol{\Gamma} = iq_x \Gamma_x$. The suffix x stands for the x component. Now

$$\Gamma_x \left\langle -\frac{\partial b}{\partial y}(\psi - \nabla^2\psi)\right\rangle_q = -\sum_k ik_y(1+k_1^2)\langle b_k \psi_{k_1}\rangle,$$

where $k_1 = q - k$.

To estimate the correlation $\langle b_k \psi_{k_1}\rangle$ we make use of a quasilinear approximation where each of these fields is generated from the other through interaction with the large-scale field. Thus we can write

$$\langle b_k \psi_{k_1}\rangle = \langle b_k \delta\psi_{k_1}\rangle + \langle \delta b_k \psi_{k_1}\rangle,$$

where it is understood that $\delta\psi_{k_1}$ is the magnetic perturbation in the plane arising as a result of turbulent stretching of the mean magnetic field by the electron flow velocity $\hat{z} \times \mathbf{k} b_k$; and δb_k is the perturbation in electron flow (viz. $\hat{z} \times \mathbf{k}\delta b_k$) arising from the Lorentz force $\hat{z}\mathbf{k}_1^2 \psi_{k_1} \times \hat{y} q_x \psi_q$. It should be noted here that the first term corresponds to that derived from a kinematic treatment, wherein the response of the magnetic field on the flow is not considered. The second term takes account of the back reaction of the magnetic field on the electron velocity. Thus, dropping the second term would be tantamount to a purely kinematic approximation. We will now show that the second term leads to a significant suppression of the estimates of diffusivity obtained purely from the kinematic treatment. The equations for δb_k and $\delta\psi_{k_1}$ are

$$(1+k_1^2)(-i\omega_k + \delta\omega_k)\delta\psi_{k_1} = -\eta k_1^2 \delta\psi_{k_1} - ik_y b_{-k} iq_x (1+q^2)\psi_q$$

and

$$(1+k^2)(-i\omega_k + \delta\omega_k)\delta b_k = -\eta k^2 \delta b_k - ik_{y1}(k_1^2 - q^2)\psi_{-k_1} iq_x \psi_q.$$

Here ω represents the linear frequency and $\delta\omega$ stands for the eddy decorrelation effect arising from coherent mode coupling. Substituting the above expression for δb_k and $\delta\psi_{k_1}$, we obtain the following expression for the nonlinear flux:

$$\Gamma_x = -\sum_k (\tau_k(k_y^2|b_k|^2 - k_{1y}^2 k_1^2 |\psi_{k_1}|^2)) iq_x \psi_q, \tag{6}$$

where

$$\tau_k = \frac{1}{(1+k^2)(-i\omega_k + \delta\omega_k) + \eta k^2}.$$

Here τ_k represents the spectral correlation time for turbulent fields. We have assumed that turbulent scales are much longer compared to electron skin depth (i.e., $k \ll 1$) in the above derivation. The evolution equation for ψ_q under the approximation $q \ll k \ll 1$ can then be written as

$$\frac{d\psi_q}{dt} = -q_x^2 \left[\sum_k \tau_k k_y^2 (|b_k|^2 - k^2 |\psi_k|^2) \right] \psi_q - \eta q_x^2 \psi_q . \quad (7)$$

The factor inside the square bracket on the right-hand side of the above equation represents the turbulent contribution to diffusivity. It is made up of two parts. The first part, depending on $k_y^2 |b_k|^2$, represents the kinematic contribution and the second part arises as a result of small scale turbulence in the poloidal component of the magnetic field. It is clear that turbulence in the poloidal component of the magnetic field contributes towards suppressing magnetic field diffusivity. It should be noted here that for complete whistlerization, spectral components of the two fields would be related as $b_k = \pm k \psi_k$, for which turbulent diffusivity vanishes exactly. For this extreme case, diffusion of ψ_q is determined by resistivity alone. It appears, then, that understanding of the question of whistlerization of the spectrum in the turbulent state is of paramount importance. We will study this issue in the next section.

We rewrite Eq. (7) as

$$\begin{aligned} \frac{d\psi_q}{dt} &= -q_x^2 \sum_k \tau_k (\langle v_x^2 \rangle_k - k^2 \langle \tilde{B}_x^2 \rangle_k) \psi_q - \eta q_x^2 \psi_q \\ &= -\frac{q_x^2}{2} \sum_k \tau_k (\langle v^2 \rangle_k - k^2 \langle \tilde{B}^2 \rangle_k) \psi_q - \eta q_x^2 \psi_q . \end{aligned} \quad (8)$$

In the above expression $\tilde{B}_x$ is the x component of the turbulent field. In writing the second equality, we have assumed that turbulence is isotropic. Thus we can write

$$\beta = \sum_k \frac{\tau_k}{2} (\langle v^2 \rangle_k - k^2 \langle (\Delta \psi)^2 \rangle_k) + \eta .$$

The kinematic diffusivity β_0 would be just $\beta_0 = \Sigma_k \tau_k v_k^2/2 + \eta$, dependent on turbulent velocity alone. We can then express β in terms of kinematic diffusivity, as $\beta = \beta_0 - \Sigma_k \tau_k k^2 \langle (\nabla \psi)^2 \rangle_k /2$. Following Gruzinov *et al.* we assume an equivalence of correlation times (i.e., assume $\tau_k = \tau$ for each mode) and write $\beta = \beta_0 - \tau \langle k^2 \rangle \langle (\nabla \psi)^2 \rangle /2$. To estimate $\langle (\nabla \psi)^2 \rangle$ we use stationarity of mean square magnetic potential. This can be justified on the basis of the inverse cascade property of the magnetic mean square potential. At longer scales, the dissipation due to resistivity is small and the assumption of stationarity of the mean square potential is reasonably good. We multiply Eq. (2) by ψ and take on ensemble average. This yields

$$\left\langle \psi \frac{d\psi}{dt} \right\rangle = \frac{1}{2} \left\langle \frac{d\psi^2}{dt} \right\rangle = 0,$$

$$\langle \psi \hat{z} \times \nabla b \cdot \nabla \psi \rangle = \tfrac{1}{2} \nabla \cdot \langle \hat{z} \times \nabla b \psi^2 \rangle = 0.$$

We thus obtain

$$\eta \langle (\nabla \psi)^2 \rangle = B_0 \left\langle \psi \frac{\partial b}{\partial y} \right\rangle = \beta B_0^2 .$$

Substituting for $\langle (\nabla \psi)^2 \rangle$ and writing $\tau/2$ as $\beta_0 / \langle v^2 \rangle = \beta_0 / \langle (\nabla b)^2 \rangle$ we obtain

$$\beta = \frac{\beta_0}{1 + \langle k^2 \rangle \beta_0 B_0^2 / \eta \langle (\nabla b)^2 \rangle} = \frac{\beta_0}{1 + \mathrm{R}_m \langle k^2 \rangle B_0^2 / \langle v^2 \rangle} . \quad (9)$$

Here R_m is the magnetic Reynold's number. It is clear that for $\mathrm{R}_m \gg 1$, suppression of magnetic field diffusivity occurs even when the turbulent velocity is larger than the effective whistler speed in the presence of B_0.

IV. WHISTLERIZATION

We have observed in an earlier section that for a turbulent state which is a collection of whistlers alone, the effective turbulent diffusivity goes to zero. Thus it is of significance to understand the whistlerization of turbulent spectra. This is identical to studying the question of Alfvénization in the context of the MHD model. It is interesting to note, however, that in the MHD model Alfvénization leads to an equipartition between the magnetic and fluid energies. However, there can be no equipartition between magnetic and kinetic energies as a result of the whistlerization of the spectrum. Here, the dominance of magnetic or kinetic energies is dependent on whether the typical scales of the turbulence are longer or shorter than the electron skin depth, respectively. In this paper we have concentrated on the case where turbulent scales are much longer compared to the electron skin depth. Thus the total energy is predominantly magnetic. Whistlerization of the spectrum then basically leads to an equipartition between poloidal and axial field energies.

We seek to understand the question of whistlerization by carrying out numerical simulation. We evolve the two fields ψ and b by Eq. (2) and Eq. (3), respectively, using a fully dealiased pseudospectral scheme. In this scheme, fields b and ψ are Fourier decomposed. Each of the Fourier modes are then evolved, the linear part exactly, whereas the nonlinear terms are calculated in real space and then Fourier transformed in k space. This requires going back and forth in real and k space at each time step. Fast Fourier transform (FFT) routines were used to go back and forth in real and k space at each time integration. Time stepping is done using a predictor corrector with a midpoint leap frog scheme. Simulation was carried out with a resolution of 128×128 modes, as well as at a higher resolution of 256×256 modes. The initial spectrum of two fields b and ψ was chosen to be concentrated on a band of scales and their phases were taken to be random. The two fields were chosen to be entirely uncorrelated to begin with.

In Fig. 1 we show a plot $|b_k|$ vs. $|k\psi_k|$ for the initial spectrum. It is clear from the figure that the initial spectrum is totally different from a spectrum of whistler waves, which in turn would have shown up in the figure as a straight line passing through the origin with the unit slope basically depicting the relationship $|b_k| = |k\psi_k|$ being obeyed. In Figs. 2 and 3 we plot for the evolved spectrum $|b_k|$ vs. $|k\psi_k|$ for $B_0 = 0$ and 0.5, respectively. It is clear that most of the points now cluster close to the origin. It is suggestive, when contrasted with the initial condition of Fig. 1, that the modes are trying to acquire a whistler wave relationship. The scatter in

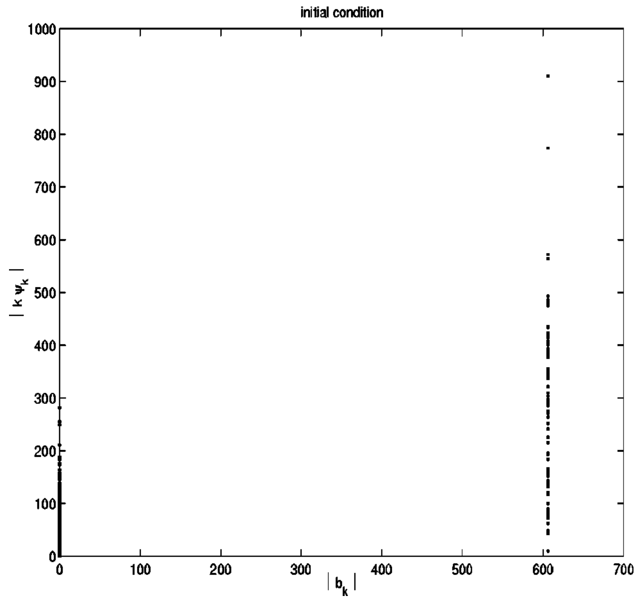

FIG. 1. Plot of $|b_k|$ vs. $|k\psi_k|$ for the initial spectrum.

the plot indicates that both eddies and whistlers constitute the final state. Thus a quantitative assessment of the turbulent state as regards whistlerization of the spectra is required. For this purpose we introduce a variable

$$w_k = \frac{abs(|b_k|^2 - k^2|\psi_k|^2)}{(|b_k|^2 + k^2|\psi_k|^2)}, \tag{10}$$

which essentially indicates the fractional deviation of k_{th} mode from being whistlerized. In Table I we list the fraction of modes in the spectrum for which w_k is within certain percentage.

It is clear from Table I that the initial state had zero fraction of modes having deviations, w_k even up to 10%; in the final state, a reasonable fraction of modes acquire whistlerization within a certain percentage of deviation, as measured by the parameter w_k. We also introduce an integral quantity signifying overall whislerization as $w = \int w_k\, dk / \int dk$. For a completely whistlerized spectrum, the variable w would take a value of 0, and the maximum value that w can have is unity. For our initial spectrum $w = 0.9957$, after evolution (i) for $B_0 = 0$ (corresponding to Fig. 1), $w = 0.5020$, and (ii) for $B_0 = 0.5$ (Fig. 2) $w = 0.4912$. More detailed studies of this kind, addressing the evolution of whislerization with time (e.g., by studing how w evolves with time), its dependence on external magnetic field, etc., will be presented in a subsequent publication.[12] The question of Alfvénization of the spectrum in the context of MHD is also being pursued along similar lines and will be presented elsewhere.

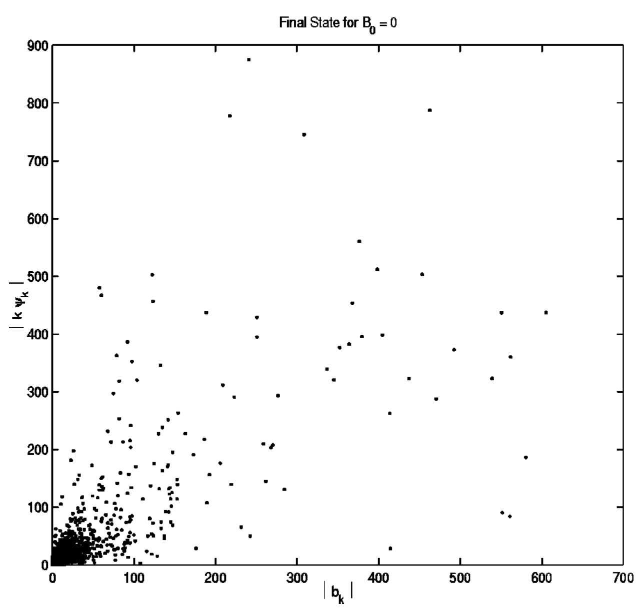

FIG. 2. Plot of $|b_k|$ vs. $|k\psi_k|$ for the evolved spectrum when the external field $B_0 = 0$.

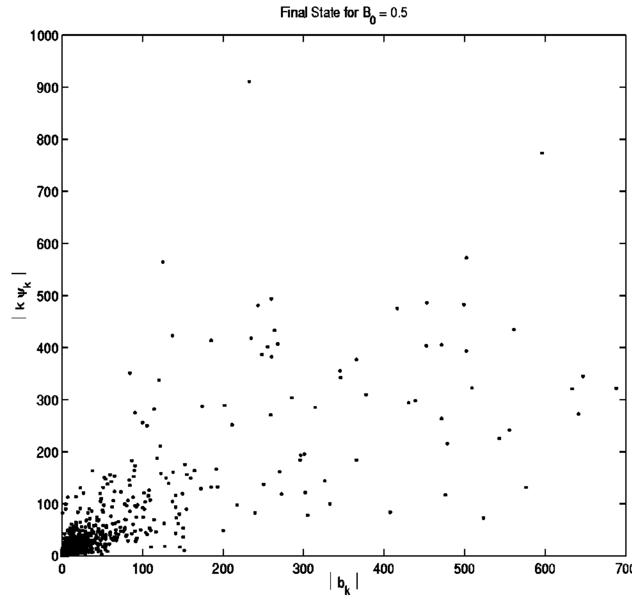

FIG. 3. Plot of $|b_k|$ vs. $|k\psi_k|$ for the evolved spectrum when the external field $B_0 = 0.5$.

It is clear from our studies that whistlerization of spectrum is not complete. Random eddies are also present in the evolved spectrum. This deviation from the whistler wave relationship contributes towards the residual effective turbulent diffusivity of magnetic field lines. In the next section we will carry out a numerical study to determine the diffusivity of the magnetic field in the presence of turbulence.

TABLE I. Quantitative whistlerization.

	Fraction of modes whistlerized		
Permissible % deviation	Initial condition	Evolved state $B_0 = 0$	Evolved state $B_0 = 0.5$
2.5	0	0.028	0.031
5	0	0.053	0.054
7.5	0	0.077	0.080
10	0	0.101	0.102

V. NUMERICAL RESULTS ON DIFFUSION

We saw in Sec. III that the final expression of effective diffusivity that we obtained was based on the fact that effective correlation times of interacting modes were ultimately the same for each of them. Whether this really happens can only be verified by a fully nonlinear numerical simulation. We have carried out a set of numerical studies to investigate the question of magnetic diffusivity. We observe that the results of our investigation agree with the expression that we have obtained earlier, thereby suggesting that the ansatz of the local equivalence of the correlation time is indeed correct.

The numerical scheme is the same as outlined in the last section. However, in addition to evolving the two fields b and ψ, a number of tracer particles ($N=1600$) were placed in the two-dimensional spatial $x-y$ region of integration. The particles were initially placed uniformly in $x-y$ plane, and were then evolved using the Lagrangian electron velocity at their location (viz. $\hat{z}\times\nabla b$). Since magnetic field lines are tied to electron flow velocity, the behavior of magnetic field diffusivity can be discerned from diffusion of these particles. Thus the averaged mean square displacement of these particles is used as a measure of magnetic diffusivity [e.g., $\beta=d\langle(\delta x)^2\rangle/dt$]. This method of evaluating the tracer particle diffusivity to study the diffusion of magnetic fields in two dimensions has been adopted by Cattaneo in the context of the MHD model.[11]

It is clear that for $\eta\neq 0$ and an initial distribution of power with random phases in various modes for the two fields b and ψ, Eqs. (2) and (3) represent the case of ''decaying'' EMHD turbulence. We refrain from using a random stirring force to achieve the stationary state, as this might lead to the particle displacement being dependent on the characteristics of the random stirrer. We will here investigate the case of decaying turbulence and we will present results in the regime where variations can be considered as slow i.e., we treat the problem in the quasistatic limit.

The derivation of our main Eq. (9) for suppression of magnetic field diffusivity was premised on the notion of the stationarity of the mean square magnetic potential. As discussed earlier, the cascade of the mean square magnetic potential towards longer scales ensures attaining such a state. This can be clearly seen in Fig. 4, which shows evolution of the mean square magnetic potential with time. It is clear that the percentage variation in $\int\psi^2\,dx\,dy$ is small after $t=200$. For the purpose of our calculations, we have restricted all our numerical runs to the region where the percentage variations in $\int\psi^2\,dx\,dy$ is below 3%.

In Fig. 5 we show the mean square displacement of tracer particles with time. The thick line indicated by the label ''kinematic'' essentially corresponds to the displacement when the uniform magnetic field in y direction B_0 is chosen to be zero. We will designate the slope of this curve as β_{kin}, the kinematic diffusivity. The other two lines essentially correspond to the longitudinal and transverse displacement in the presence of a uniform magnetic field $B_0=1$ along the y direction. It is clear from the figure that the slope of the kinematic curve is larger than the other two curves

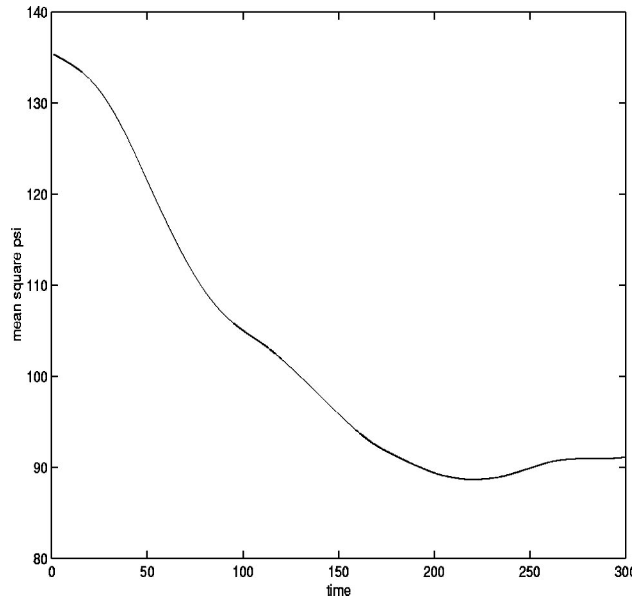

FIG. 4. Evolution of mean square magnetic potential.

which correspond to the displacement for finite B_0. This clearly indicates that the presence of B_0 suppresses diffusivity, a conclusion we arrived at earlier in the last section. However, longitudinal displacements of tracer particles are larger compared to their transverse displacement, suggesting that the assumption of isotropic turbulence is not valid in the presence of uniform magnetic field. There has been indications in earlier works, both in MHD[13] as well as in EMHD,[12] that the presence of strong magnetic field results in anisotropy of the spectrum. Our results showing distinct values for longitudinal and transverse diffusivity is another piece of

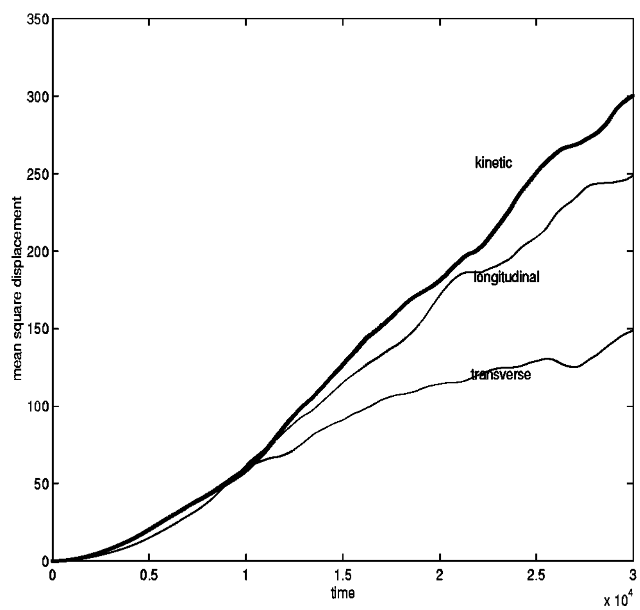

FIG. 5. Mean square displacement of the tracer particles with time is shown, thick lines (kinematic) shows the displacement in the absence of any external field. The other two lines indicated by ''longitudinal'' and the ''transverse'' show the mean square displacement of the tracer particles along and across the external magnetic field $B_0=1$.

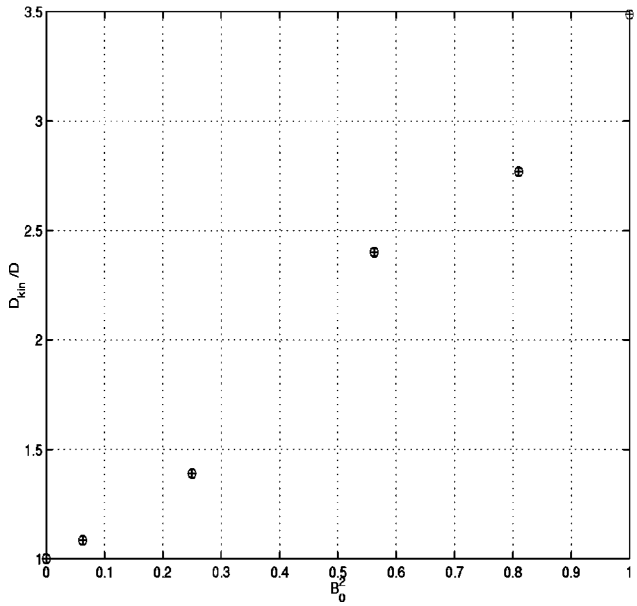

FIG. 6. A plot of $\beta_{\rm kin}/\beta$ vs. B_0^2.

evidence for anisotropic turbulence in the presence of an external magnetic field.

We next investigate the question whether suppression of diffusivity with an increasing magnetic field is indeed given by the kind of expression [Eq. (9)] that we have obtained in the earlier section. For this purpose we carry out several numerical runs with varying strength of the magnetic field B_0. The diffusivity β for each case is then given by the slope of the displacement of tracer particles. It is clear from Fig. 5 that the curve is jagged, essentially signifying that β, the diffusivity estimated from the slope of such a curve, is a statistical quantity. We take a time average given by

$$\beta(t_2-t_1)=\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\beta(t)\,dt.$$

The choice of t_2-t_1 is such that, in this duration, turbulence can essentially be treated as quasistationary. The averaging procedure eliminates statistical fluctuation in the estimate of diffusity and it is observed that with varying t_2 the value of the slope asymptotes to a constant for each case.

In Fig. 6 the y axis represents $\beta_{\rm kin}/\beta$ and along the x axis we vary B_0^2. It is clear from the plot that data points nicely fit a straight line, as our analytical expression predicts.

VI. DISCUSSION

There are two important results of our present work. First, we have been able to show that the turbulent EMHD state shows tendencies towards whistlerization. The spectrum is only partially whistlerized, suggesting that both eddies and randomly interacting whistlers constitute the turbulent state. Second, we have carried out studies to understand the diffusion of the long-scale magnetic field in the context of electron magnetohydrodynamics. We have shown that the effective diffusivity due to turbulence in the electron flow velocity is suppressed in the presence of a small-scale turbulence of the magnetic field. For complete whistlerization the turbulent diffusivity vanishes. However, since the turbulent state is only partially whistlerized, the effective diffusivity does not vanish, it only becomes suppressed from pure kinematic estimates. We have confirmed these results numerically.

The problem of diffusion of magnetic field is of great interest, as it provides a mechanism for the reconnection of magnetic field lines, which is thought to underlie an understanding of the rapid release of energy in several solar and astrophysical contexts. The resistive diffusion turns out to be too small to explain the large amount of energy released. This had initiated efforts towards understanding the phenomenon of turbulent diffusivity of magnetic field lines. Earlier attempts on this were based on the magnetohydrodynamic approximation. However, it was shown theoretically by Gruzinov *et al.*[10] and numerically by Cattaneo[11] that the value of turbulent diffusivity is suppressed in the presence of turbulence in a small-scale magnetic field. Recently, attempts to understand the reconnection phenomenon in the context of electron magnetohydrodynamics were made.[8,6] Our work in this context becomes relevant, as we have shown here that the naive quasilinear estimates do not provide a complete picture. The effective diffusivity is suppressed in the presence of turbulent magnetic fields, with whistlerization of the spectrum playing an important role in this regard.

Another issue that we would like to point out in this regard is about the role of whistlers in EMHD turbulence. Some recent studies on EMHD turbulence categorically rule out the presence of the whistler effect in determining the energy transfer rate on the basis of numerically observed scaling of the power spectrum.[9] We have, on the other hand, shown here that there is a tendency towards whistlerization of the turbulent spectra which directly influences the effective diffusivity of magnetic field lines. Invoking the Prandtl mixing length argument, which relates the transfer rate to the effective diffusivity, the question of the whistler effect being present or not remains debatable. Moreover, we also have evidence of anisotropization of the turbulent spectrum in the presence of external magnetic field,[12] which further points towards a subtle role of whistlers in governing EMHD turbulence.

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